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Coepi maps and generalizations of the Hopf extension theorem

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Abstract

The Hopf extension theorem states that a map on the unit sphere in $\mathbb{R}^n$ is essential (i.e., each continuous extension to the unit ball has a zero) if and only if it has nonzero rotation (degree). We formulate and prove a corresponding result for coincidence points of condensing pairs of maps in infinite-dimensional spaces. To this end, a theory of coepi maps is introduced which in some sense is dual to the known theory of 0-epi maps. Also a uniqueness result for the coincidence index is obtained which provides a way to effectively calculate the index.

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1. Coepi maps

Let X and Y be Banach spaces, and $\Omega \subseteq X$ be open and bounded. Recall that a continuous map $F:\overline{\Omega} \to Y$ is called 0-*epi* [19,30] if $F(x) \neq 0$ on $\partial\Omega$ and the equation $F(x) = \varphi(x)$ has a solution for each compact continuous map $\varphi:\overline{\Omega} \to Y$ with $\varphi|_{\partial\Omega} = 0$. (Sometimes such maps are called *essential*, see, e.g., [4,25]. The reason for this notational confusion is that this class of maps was introduced independently by M. Furi/M. Martelli/A. Vignoli and A. Granas.)

It turns out that 0-epi maps are precisely those maps which have a zero which is stable under admissible compact homotopic perturbations. On the other hand, 0-epi maps have many properties in common with maps with nonzero degree like homotopy invariance,

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normalization, stability under restriction of Ω, and boundary dependence. Hence, one might consider 0-epi maps as a homotopic analogue to degree theory (which in turn might be considered as an application of homology theory). Since the famous Hopf theorems [26–28] (see also, e.g., [2,13,29]) establish a connection between homotopy and homology theory, one might suspect that it is possible to find a link between the theory of 0-epi maps and degree theory.

Indeed, in the paper [20] it was proved that if $F = \mathrm{id} - \varphi$ where φ is a so-called strictly condensing map, then F is 0-epi if and only if it has nonzero degree on some component of Ω. It is the purpose of this paper to generalize this result when id is replaced by a so-called Vietoris map. In this case, we may of course not speak of the classical Nussbaum–Sadovskiĭ degree as before. Instead, we use the Kryszewski–Górniewicz index for so-called morphisms [22,35,36] (in case of compact φ) respectively the coincidence index from [5, 43] (for condensing φ).

However, it appears that the above definition of "0-epi" is not appropriate for our needs. We need a notion which is more related to cohomotopy theory. For this reason, we call the corresponding maps *coepi*. Theorems 3 and 12 will show that this is indeed the "right" notion for a comparison with the Kryszewski–Górniewicz index. The following definition of coepi maps is very similar to 0-epi maps. The main difference is that instead of considering sets $\Omega \subseteq X$, we consider sets of the form $F^{-1}(\Omega)$ with $\Omega \subseteq Y$.

Let X and Y be topological spaces, and $F : X \to Y$. For $M \subseteq Y$, we put

$$F^{-1}(M) := \{x \in X\colon\ F(x) \in M\}.$$

For a map $\varphi : D \to Y$ with $D \subseteq X$ and $M \subseteq Y$, we define the coincidence point image

$$\mathrm{Coin}(F, \varphi, M) := F\bigl(\{x \in D\colon\ F(x) = \varphi(x) \in M\}\bigr) = \{F(x)\colon\ F(x) = \varphi(x)\} \cap M.$$

Similarly, we put for maps $h : I \times D \to Y$ also

$$\mathrm{Coin}(F, h, M) := F\bigl(\{x\colon\ F(x) \in h\bigl(I \times \{x\}\bigr) \cap M\}\bigr) = \bigcup_{t\in I} \mathrm{Coin}\bigl(F, h(t,\cdot), M\bigr).$$

As usual, we call a set *relatively compact* if its closure is compact.

Definition 1. Let X and Y be topological spaces, $F : X \to Y$, and $\Omega \subseteq Y$. Let $f : D \to Y$ with $F^{-1}(\partial\Omega) \subseteq D \subseteq X$. Then we call F

(a) *f-coadmissible* (on Ω) if $\mathrm{Coin}(F, f, \partial\Omega) = \emptyset$.
(b) *f-coepi* (on Ω with respect to Y) if F is f-coadmissible, and if there is some continuous map $\varphi : F^{-1}(\overline{\Omega}) \to Y$ with relatively compact range and $\varphi(x) = f(x)$ on $F^{-1}(\partial\Omega)$, and moreover, if for each such map $\mathrm{Coin}(F, \varphi, \Omega) \neq \emptyset$.

The above definition is only useful if f has relatively compact range. For noncompact maps, we consider only a more special situation.

Let X be a topological space, Y be a closed subset of some topological vector space Z, and $F : X \to Y$. Let $\Omega \subseteq Y$ and $f : D \to Y$ with $F^{-1}(\overline{\Omega}) \subseteq D \subseteq X$.

Definition 2. In the above situation, we call F *convexly f-coepi* (on Ω with respect to (Y, Z)) if it is f-coadmissible on Ω, and if there is some continuous $\varphi : F^{-1}(\overline{\Omega}) \to Y$

with $\varphi(x) = f(x)$ on $F^{-1}(\partial\Omega)$ for which $\overline{\operatorname{conv}}((\varphi - f)(F^{-1}(\Omega))) \subseteq Z$ is compact, and moreover, if for each such map $\operatorname{Coin}(F, \varphi, \Omega) \neq \emptyset$.

The main difference to Definition 1 is that the compactness assumption for φ is replaced by a compactness assumption for the difference $\varphi - f$.

Proposition 1. *Assume that in the above situation f is continuous with a relatively compact range. If F is f-coepi on Ω, then F is convexly f-coepi on Ω. The converse holds if Z is a Fréchet space.*

We call in the above situation a map $G : F^{-1}(\overline{\Omega}) \to Y$ an f-coadmissible (convexly) compact homotopic perturbation of F if there is some homotopy $H : [0, 1] \times F^{-1}(\overline{\Omega}) \to Z$ with $G = F + H(1, \cdot)$ such that the following holds.

(a) The range R of H is relatively compact (respectively $\overline{\operatorname{conv}}\, R$ is compact).
(b) $R + Y \subseteq Y$.
(c) $F(x) - H(t, x) \subseteq Y$ and $H(0, \cdot) = 0$.
(d) $F(x) - H(t, x) \neq f(x)$ on $[0, 1] \times F^{-1}(\partial\Omega)$.

The proof of the following two results is similar to [6].

Proposition 2 (Homotopic stability). *Let X be normal and F be f-coadmissible. If F is* (*convexly*) *f-coepi, then for each G as above and each continuous map $\varphi : F^{-1}(\overline{\Omega}) \to Y$ with $\varphi(x) = f(x)$ on $F^{-1}(\partial\Omega)$ and relatively compact range* (*respectively compact* $\overline{\operatorname{conv}}((\varphi - f)(F^{-1}(\Omega))))$ *the equation $G(x) = \varphi(x)$ has a solution in $F^{-1}(\Omega)$. Also the converse holds, if a map φ as above exists.*

Proposition 3 (Restriction property). *Let X and Y be topological spaces, and $\Omega_0 \subseteq \Omega \subseteq Y$ where Ω_0 is open in Y.*

(a) *If $f : F^{-1}(\overline{\Omega} \setminus \Omega_0) \to Y$ is continuous with relatively compact range and F is f-coepi on Ω with* $\operatorname{Coin}(F, f, \overline{\Omega} \setminus \Omega_0) = \emptyset$, *then F is f-coepi on Ω_0.*
(b) *If $f : F^{-1}(\overline{\Omega}) \to Y$ is continuous and F if convexly f-coepi on Ω with* $\operatorname{Coin}(F, f, \overline{\Omega} \setminus \Omega_0) = \emptyset$, *then F is f-coepi on Ω_0.*

Proposition 4 (Restriction property for components). *Let X, Y be topological spaces, and $F : X \to Y$ be continuous and closed. Let $\Omega \subseteq Y$ be open and such that all components of Ω are also open. Let $f : F^{-1}(\overline{\Omega}) \to Y$ be continuous such that F is f-coadmissible on Ω. Then F is f-coadmissible on each component of Ω, and the following holds.*

(a) *If f has a relatively compact range, then F is f-coepi on Ω if and only if F is f-coepi on some component of Ω.*
(b) *If Y is a closed subset of some topological vector space and* $\operatorname{Coin}(F, f, \Omega)$ *is contained in a compact set, then F is convexly f-coepi on Ω if and only if F is convexly f-coepi on some component of Ω.*

Proof of Proposition 4. One implication follows from $\partial\Omega_0 \subseteq \partial\Omega$. Conversely, let F be f-coepi. Then $C := \operatorname{Coin}(F, f, \overline{\Omega}) = \operatorname{Coin}(F, f, \Omega)$ is a compact subset of Ω. By the compactness, C is contained in the union of finitely many (open) components $\Omega_1, \ldots, \Omega_N$ of Ω. Assume that F fails to be f-coepi on each Ω_n. Then we find for each $n = 1, \ldots, N$ some continuous $\varphi_n : F^{-1}(\overline{\Omega}_n) \to Y$ with relatively compact range and with $\varphi_n(x) = f(x)$ on $F^{-1}(\partial\Omega_n)$ such that $\operatorname{Coin}(F, \varphi_n, \overline{\Omega}_n) = \emptyset$. Define $\varphi : F^{-1}(\overline{\Omega}) \to Y$ by $\varphi(x) := \varphi_n(x)$ for $x \in F^{-1}(\Omega_n)$ and $\varphi(x) := f(x)$ for $x \notin \bigcup F^{-1}(\Omega_n)$. Then $\operatorname{Coin}(F, \varphi, \overline{\Omega}) = \emptyset$, a contradiction. The proof of the second claim is analogous.

2. The compact case

The results in this section extend some results of [35] and [36]. We call a map $F : X \to Y$ in metric spaces *Vietoris* if it is continuous, surjective, and proper (i.e., preimages of compact sets are compact), and if each fibre $F^{-1}(\{y\})$ is acyclic with respect to the Čech cohomology with integer or rational coefficients. Recall that each proper map in metric spaces sends closed sets to closed sets.

Let Z be a fixed (metric) ANR [12,22]. Let X be a metric space, $Y \subseteq Z$ be an ANR, and $\Omega \subseteq Y$ be open. Let $F : X \to \overline{\Omega}$ be a Vietoris map, and $\varphi : X \to Y$ be continuous with $\operatorname{Coin}(F, \varphi, \partial\Omega) = \emptyset$. In addition, assume that $\varphi \circ F^{-1}$ is *locally compact*, i.e., that each point has a neighborhood on which this multivalued map has relatively compact range. Finally, assume that $\operatorname{Coin}(F, \varphi, \Omega)$ is relatively compact.

Theorem 1 (Kryszewski, Górniewicz). *Let Z be as above. Then one can associate to each X, F, φ, Ω, and Y as above an integer number $\operatorname{ind}_F(\varphi, \Omega, Y)$ such that the following holds.*

(a) (Normalization). *If $\varphi \circ F^{-1}$ is a single-valued map (on $\overline{\Omega}$), then $\operatorname{ind}_F(\varphi, \Omega, Y)$ is the classical fixed point index of this map.*
(b) (Coincidence point property). *If $\operatorname{ind}_F(\varphi, \Omega, Y) \neq 0$, then the coincidence equation $F(x) = \varphi(x) \in \Omega$ has a solution.*
(c) (Homotopy invariance). *If $H : [0,1] \times X \to Y$ is continuous with $\operatorname{Coin}(F, H, \partial\Omega) = \emptyset$ and such that the multivalued map $H([0,1] \times F^{-1}(\cdot))$ is locally compact and $\operatorname{Coin}(F, H, \Omega)$ is relatively compact, then*

$$\operatorname{ind}_F\bigl(H(0,\cdot), \Omega, Y\bigr) = \operatorname{ind}_F\bigl(H(1,\cdot), \Omega, Y\bigr).$$

(d) (Additivity). *If $\Omega_1, \Omega_2 \subseteq \Omega$ are disjoint and open in Y with $\operatorname{Coin}(F, \varphi, \overline{\Omega}) \subseteq \Omega_1 \cup \Omega_2$, then*

$$\operatorname{ind}_F(\varphi, \Omega, Y) = \operatorname{ind}_F(\varphi, \Omega_1, Y) + \operatorname{ind}_F(\varphi, \Omega_2, Y).$$

(e) (Permanence). *If $Y_0 \subseteq Y$ is an ANR which contains the closure of the range of φ, then*

$$\operatorname{ind}_F(\varphi, \Omega, Y) = \operatorname{ind}_F(\varphi, \Omega \cap Y_0, Y_0).$$

Although related results exist, Theorem 1 is new: A similar index is given by the index for compositions of multivalued acyclic maps which is obtained by the method of simplicial approximations [17,39]; see also [18,46] or [22, Section 50–53]. However, that index requires slightly more restrictively that $F^{-1}(\{y\})$ be acyclic with respect to the Čech cohomology with coefficients in a *field* $\mathbb{K}$ (e.g., $\mathbb{K} = \mathbb{Q}$). Moreover, it is only clear from the construction that the index then assumes values in $\mathbb{K}$. It is unknown whether in case $\mathbb{K} = \mathbb{Q}$ that index actually assumes only integer numbers. Using homotopic methods, a different approach was used in [35] (see also [22, Theorem (47.8)]) to obtain an index whose values are integer numbers but for which it was required instead that the fibres $F^{-1}(\{y\})$ have uniformly bounded (finite) covering dimension (be aware that this condition was mistakenly forgotten in [22]). Without this condition, it is not known whether that index is topologically invariant with respect to Y.

Proof of Theorem 1. First, let Z be a normed space. Since Y is a neighborhood retract of Z, one may use the analogous definition as in [35, Section 2] for the index with the difference that one always uses Z as the embedding space with the inclusion as the embedding. With this modification, one can show analogously to the proof of [35, Theorem (2.3)] that the index is well-defined (and has the required properties): The topological invariance with respect to Y is not needed for the proof, because one can choose $h = \mathrm{id}$ in that proof.

For a general ANR Z, we use the Arens–Eells embedding theorem [7] to observe that Z is a closed subspace of some normed space $\widehat{Z}$. We choose some index as above corresponding to $\widehat{Z}$. The restriction of this index to those ANRs Y which are contained in Z has the required properties (although we do not know whether this index is independent of our choice of $\widehat{Z}$ or of the embedding of Z into $\widehat{Z}$). □

We note that the index obtained in the previous proof is actually an index for morphisms in the sense of [34,35]. In particular, this index has also some sort of homotopic stability with respect to F. However, we shall not make use of this fact. We do not know whether the obtained index $\mathrm{ind}_F(\varphi, \Omega, Y)$ is unique or can at least be defined independently of the choice of Z, in general. However, under a mild additional assumption on F, we can prove this uniqueness. In particular, under this assumption, the index is actually the same as the index obtained by the above mentioned method of chain approximations. Hence, in this special case, the index from [39], [18,46] or [22, Section 50–53] actually assumes integer values. Under the first of the following assumptions this was already observed in [36] (the assumption was actually slightly more restrictive in [36]).

Definition 3. Let X be a metric space, and M be a subset of some locally convex metrizable space Y. Then we call $F : X \to M$ a *Vietoris* map*, if F is continuous, surjective, and proper, and one of the following holds.

(a) Each fibre $F^{-1}(\{y\})$ is acyclic with respect to the Čech cohomology with integer or rational coefficients, and additionally for each compact subset $A \subseteq M$ which is contained in a finite-dimensional subspace of Y the relation

$$\sup\bigl\{\dim F^{-1}\bigl(\{y\}\bigr):\ y \in A\bigr\} < \infty \tag{1}$$

holds. Here, dim denotes the covering dimension of the set.

(b) Alternatively, assume that each fibre $M = F^{-1}(\{y\})$ is a so-called R_δ-*set*, i.e., it may be represented as the intersection of a decreasing countable sequence of compact contractible metric spaces.

Each Vietoris* map is a Vietoris map (because R_δ-sets are acyclic) and the converse holds if X has finite dimension.

Theorem 2 (Uniqueness of ind_F). *Let $Y \subseteq Z$ be a locally convex metrizable vector space, and $F : X \to Y$ be a Vietoris* map. Then $\mathrm{ind}_F(\varphi, \Omega, Y_0)$ is uniquely determined by the properties of Theorem* 1 (*required only for this fixed F and ANRs $Y_0 \subseteq Y$*). *In particular, ind_F is independent of the choice of Z. The theorem holds also if the additivity is replaced by the* excision property: *If $\Omega_0 \subseteq \Omega$ is open in Y with* $\mathrm{Coin}(F, \varphi, \overline{\Omega}) \subseteq \Omega_0$, *then*

$$\mathrm{ind}_F(\varphi, \Omega, Y) = \mathrm{ind}_F(\varphi, \Omega_0, Y).$$

We emphasize that Theorem 2 is new, since in contrast to [22,35], we keep F fixed, i.e., no sort of homotopy invariance of ind_F with respect to F is assumed in Theorem 2. In the particular case of R_δ fibres, Theorem 2 implies that our index corresponds to the fixed point index for the multivalued map $\varphi \circ F^{-1}$ which is based on graph-approximation by single-valued maps (see, e.g., [3,8,23,24]).

Theorem 3 (Hopf extension theorem for the compact case). *Let Y be a locally convex metrizable vector space, $\Omega \subseteq Y$ be open, X be a metric space, and $F : X \to \overline{\Omega}$ be a Vietoris* map. Then for any continuous map $\varphi : X \to Y$ with relatively compact range the following statements are equivalent.*

(a) *F is φ-coepi on Ω.*
(b) *There is some component Ω_0 of Ω with* $\mathrm{ind}_F(\varphi, \Omega_0, Y) \neq 0$.

It suffices to prove Theorem 3 for the case that Ω is connected: Indeed, the components of an open set Ω in a normed space Y are always path-components and open. The implication (b) $\Rightarrow$ (a) thus is a straightforward consequence of the boundary dependence of the index which in turn follows from the homotopy invariance. The rest of this section is devoted to the proof of the reverse implication which is a generalization of the Hopf extension theorem. Our main tool is the following result.

Theorem 4 (Kryszewski). *Let X be a compact metric space, A be a compact subset of a finite-dimensional* (*normed*) *space Y, and $F : X \to A$ be a Vietoris* map. Then for each continuous $\varphi : X \to Y$ with* $\mathrm{Coin}(F, \varphi, A) = \emptyset$, *there is a* (*fixed point free*) *continuous map $G : A \to Y$ and some homotopy $H : [0, 1] \times X \to Y$ with $H(0, \cdot) = \varphi$ and $H(1, \cdot) = G \circ F$ such that* $\mathrm{Coin}(F, H, A) = \emptyset$.

Replacing φ by $F - \varphi$, G by $\mathrm{id} - G$, and H by $F - H$, we see that Theorem 4 becomes:

Theorem 5 (Kryszewski). *Let X be a compact metric space, A be a compact subset of a finite-dimensional* (*normed*) *space Y, and $F : X \to A$ be a Vietoris* map. Then for each* *continuous $\varphi : X \to Y \setminus \{0\}$ there is a continuous map $G : A \to Y \setminus \{0\}$ and some homotopy $H : [0,1] \times X \to Y \setminus \{0\}$ with $H(0,\cdot) = \varphi$ and $H(1,\cdot) = G \circ F$.*

For the case that F has R_δ fibres, Theorem 5 is a special case of [36, Theorem 2.19(i)] (see also [16] or [15, Theorem B]). If (1) holds, then Theorem 5 is a special case of [34, Corollary (2.12)], or also of [36, Theorem 2.17(i)] if one observes that in view of [36, Remark 4.3(ii)] (i.e., by [40]) the space X has actually finite dimension in this case. Under assumption (1), Theorem 4 is also a special case of [35, Lemma (1.3)] (see also [22, Lemma (47.1)]). The above theorems do not hold for arbitrary Vietoris maps, as was observed in [34, Example (2.13)(i)] and in [36, Example 2.22].

Lemma 1 (Schauder projection). *Let Y be a metric locally convex space* (*with a translation invariant metric d*). *Let $R \subseteq Y$ be compact, and $\varepsilon > 0$. Then there is some finite-dimensional subspace $\widehat{Y} \subseteq Y$ and a continuous map $\pi : R \to \widehat{Y} \cap \operatorname{conv} R$ with $d(\pi(y), y) < \varepsilon$ on R.*

Lemma 1 follows from a modification of the standard construction in a normed space.

Lemma 2. *Let Y be a locally convex metrizable space, $\Omega \subseteq Y$ be open and connected, and $C \subseteq \Omega$ be compact. Then there is some connected open $\Omega_0 \supseteq C$ with $\overline{\Omega}_0 \subseteq \Omega$ and some finite set $S \subseteq \Omega_0$ such that for each subspace $\widehat{Y} \supseteq Y$ with $\widehat{Y} \supseteq S$ the intersection $\Omega_0 \cap \widehat{Y}$ is connected. If Y is normed, Ω_0 may be chosen bounded.*

The following result is the required generalization of [35, Corollary (1.12)(i)].

Theorem 6. *Let Y be a locally convex metrizable space, and $\Omega \subseteq Y$ be open and connected. Let X be a metric space, $F : X \to \overline{\Omega}$ be a Vietoris* map, and $\varphi : X \to Y$ be continuous with relatively compact range and such that* $\operatorname{Coin}(F, \varphi, \partial\Omega) = \emptyset$ *and*

$$\operatorname{ind}_F(\varphi, \Omega, Y) = 0.$$

Then F is not φ-coepi on Ω.

Proof. We first assume that Y has finite dimension. Since $C := \operatorname{Coin}(F, \varphi, \overline{\Omega})$ is compact, we find by Lemma 2 some connected open bounded set $\Omega_0 \supseteq C$ with $\overline{\Omega}_0 \subseteq \Omega$, and so

$$\operatorname{ind}_F(\varphi, \Omega_0, Y) = \operatorname{ind}_F(\varphi, \Omega, Y) = 0.$$

Choose some open bounded $\Omega_1 \subseteq \Omega$ with $\overline{\Omega}_0 \subseteq \Omega_1$, and put $A := \overline{\Omega}_1 \setminus \Omega_0$. Since the restriction of F to $\widehat{X} := F^{-1}(A)$ is a Vietoris* map $F : \widehat{X} \to A$, Theorem 4 provides a continuous map $G : A \to Y$ and a homotopy $H : [0,1] \times F^{-1}(A) \to Y$ such that $H(0,\cdot) = \varphi$, $H(1,\cdot) = G \circ F$, and $\operatorname{Coin}(F, H, A) = \emptyset$. By the Tietze–Urysohn extension theorem, we may assume that $G : \overline{\Omega}_1 \to Y$ is continuous. Similarly, we also extend H to a continuous map $H : [0,1] \times F^{-1}(\overline{\Omega}_1) \to Y$ by first putting $H(0,x) := \varphi(x)$ and $H(1,\cdot) := G(F(x))$ for $x \in F^{-1}(\overline{\Omega}_1)$, and then applying Tietze–Urysohn. The homotopy invariance of the index implies

$$\begin{aligned}\operatorname{ind}_F(G\circ F,\Omega_0,Y) &= \operatorname{ind}_F\bigl(H(1,\cdot),\Omega_0,Y\bigr)=\operatorname{ind}_F\bigl(H(0,\cdot),\Omega_0,Y\bigr)\\ &= \operatorname{ind}_F(\varphi,\Omega_0,Y)=0.\end{aligned}$$

Since $(G\circ F)\circ F^{-1}=G:\overline{\Omega}_0\to Y$ is single-valued, we thus have by the normalization property of the index that $0=\operatorname{ind}_F(G\circ F,\Omega_0,Y)=\operatorname{Deg}(\mathrm{id}-G,\Omega_0,0)$. Hence, we may apply the Hopf extension theorem (see, e.g., [32, II. §3 Section 9]) to find that there is some map $G_0:\overline{\Omega}_0\to Y$ without fixed points which coincides with G on $\partial\Omega_0$. Since $\overline{\Omega}_0\subseteq\Omega_1$ implies that the closed sets $\partial\Omega_0$ and $\partial\Omega_1$ are disjoint, we find by Urysohn's lemma a continuous function $\lambda:F^{-1}(\partial\Omega_1)\to[0,1]$ with $\lambda|_{F^{-1}(\partial\Omega_1)}=0$ and $\lambda|_{F^{-1}(\partial\Omega_0)}=1$. Then the function

$$\Phi(x):=\begin{cases}H\bigl(\lambda(x),x\bigr) & \text{if } x\in F^{-1}(A)=F^{-1}\bigl(\overline{\Omega}_1\bigr)\setminus F^{-1}(\Omega_0),\\ G_0\bigl(F(x)\bigr) & \text{if } x\in F^{-1}(\Omega_0).\end{cases}$$

is continuous on $F^{-1}(\overline{\Omega}_1)$, since for $x\in\partial F^{-1}(\Omega_0)\subseteq F^{-1}(\partial\Omega_0)$ we have $H(\lambda(x),x)=H(1,x)=G(F(x))=G_0(F(x))$. Observe that Φ has relatively compact range and satisfies $F(x)\neq\Phi(x)$ on $F^{-1}(\overline{\Omega}_1)$. Moreover, for $x\in F^{-1}(\partial\Omega_1)\supseteq\partial F^{-1}(\Omega_1)$ we have $\Phi(x)=H(0,x)=\varphi(x)$, and so putting $\Phi(x):=\varphi(x)$ for $x\in F^{-1}(\overline{\Omega}\setminus\overline{\Omega}_1)$, we have extended Φ to a continuous map satisfying $\Phi(x)=\varphi(x)$ on $F^{-1}(\partial\Omega)$ and $\operatorname{Coin}(F,\Phi,\overline{\Omega})=\emptyset$. This proves the statement for the case that Y has finite dimension.

Now we attack the proof of the general case. Since $C:=\operatorname{Coin}(F,\varphi,\overline{\Omega})$ is compact, we find by Lemma 2 some connected open set $\Omega_0\supseteq C$ with $\overline{\Omega}_0\subseteq\Omega$ and some finite set $S\subseteq\Omega_0$ such that for each subspace $\widehat{Y}\subseteq Y$ with $\widehat{Y}\supseteq S$ the intersection $\Omega_{\widehat{Y}}:=\Omega_0\cap\widehat{Y}$ is connected. The excision property of the index implies

$$\operatorname{ind}_F(\varphi,\Omega_0,Y)=\operatorname{ind}_F(\varphi,\Omega,Y)=0.$$

Let d be some translation invariant metric generating the topology of Y with convex balls, and put $A:=\overline{\Omega}\setminus\Omega_0$ and $\varepsilon:=\inf\{d(\varphi(x),F(x)):\ x\in F^{-1}(A)\}$. The properness of F implies $\varepsilon>0$. Since $R:=\overline{\varphi(X)}$ is compact, we find by Lemma 1 a finite-dimensional subspace $\widehat{Y}\subseteq Y$ and a continuous map $\pi:R\to\widehat{Y}$ such that $\hat{\varphi}:=\pi\circ\varphi$ satisfies $d(\varphi(x),\hat{\varphi}(x))<\varepsilon$. Enlarging $\widehat{Y}$ if necessary, we may assume $\widehat{Y}\supseteq S$, i.e., $\Omega_{\widehat{Y}}$ is connected (and open in $\widehat{Y}$). Since balls are convex, we may conclude that $d(\varphi(x),h(\lambda,x))<\varepsilon$ for $\lambda\in[0,1]$ where $h(\lambda,\cdot):=\lambda\hat{\varphi}+(1-\lambda)\varphi$. In particular, $F(x)\neq h(\lambda,x)$ for $x\in F^{-1}(A)$. By homotopy invariance of the index

$$\operatorname{ind}_F(\hat{\varphi},\Omega_0,Y)=\operatorname{ind}_F(\varphi,\Omega_0,Y)=0.$$

Moreover, in view of the permanence property, we also have

$$0=\operatorname{ind}_F(\hat{\varphi},\Omega_0,Y)=\operatorname{ind}_F\bigl(\hat{\varphi},\Omega_{\widehat{Y}},\widehat{Y}\bigr).$$

Since $\Omega_{\widehat{Y}}$ is connected, we may thus apply what we have proved before to find some continuous map $\psi:F^{-1}(\Omega_{\widehat{Y}})\to\widehat{Y}$ with bounded range and $F(x)\neq\psi(x)$ everywhere such that ψ coincides with $\hat{\varphi}$ on $F^{-1}(B)$ where B denotes the boundary of $\Omega_{\widehat{Y}}$ in $\widehat{Y}$. Since the subspace $\widehat{Y}$ has finite dimension, it is complete and thus closed in Y which implies $(\partial\Omega_0)\cap\overline{\Omega}_{\widehat{Y}}=\overline{\Omega}_{\widehat{Y}}\setminus\Omega_0=B$. Hence, extending ψ by $\psi(x):=\hat{\varphi}(x)$ for $x\in F^{-1}(\partial\Omega_0\setminus B)$, we have a continuous map ψ with bounded range and $F(x)\neq\psi(x)$ everywhere. By Tietze–Urysohn, we may extend ψ to a continuous function $\psi:F^{-1}(\overline{\Omega}_0)\to\widehat{Y}$ with bounded range. Since $F(x) = \psi(x)$ would imply that $F(x) \in \widehat{Y}$ and thus $x \in F^{-1}(\overline{\Omega}_0 \cap Y) \subseteq F^{-1}(\partial\Omega_0) \cup F^{-1}(\Omega_{\widehat{Y}})$, we still have $F(x) \neq \psi(x)$ everywhere. Now we continue similarly as in the first step of the proof: By Urysohn's lemma there is a continuous function $\lambda : X \to [0,1]$ with $\lambda|_{F^{-1}(\partial\Omega)} = 0$ and $\lambda|_{F^{-1}(\partial\Omega_0)} = 1$. Then

$$\Psi(x) := \begin{cases} h\big(\lambda(x), x\big) & \text{if } x \in X \setminus F^{-1}(\Omega_0), \\ \psi(x) & \text{if } x \in F^{-1}(\Omega_0) \end{cases}$$

is continuous on $F^{-1}(\overline{\Omega})$, since for $x \in \partial F^{-1}(\Omega_0) \subseteq F^{-1}(\partial\Omega_0)$ we have $h(\lambda(x), x) = h(1, x) = \hat{\varphi}(x) = \psi(x)$. Moreover, Ψ coincides with φ on $F^{-1}(\partial\Omega)$, and satisfies $F(x) \neq \Psi(x)$ on X. $\square$

Applying [21] and the Arens–Eells embedding theorem, one can prove:

Lemma 3. *Let X be a metric space, Y be an ANR, $A \subseteq X$ be closed, and $\varphi : A \to Y$ be continuous with relatively compact range. Then there is some neighborhood $U \supseteq A$ of A and a continuous extension $\varphi : U \to Y$ with relatively compact range. If Y is an AR, one may choose $U = X$.*

Proof of Theorem 2. Let $\Omega \subseteq Y_0$ be open, and $\varphi : F^{-1}(\overline{\Omega}) \to Y_0$ be continuous and such that $\varphi \circ F^{-1}$ is locally compact with compact $C := \mathrm{Coin}(F, \varphi, \overline{\Omega}) \subseteq \Omega$. Since C is compact and Y is normal, we find some open $\Omega_0 \supseteq C$ with $\overline{\Omega}_0 \subseteq \Omega$ such that $\varphi(F^{-1}(\overline{\Omega}_0))$ is relatively compact. For each index with the excision property, we must have $\mathrm{ind}_F(\varphi, \Omega_0, Y_0) = \mathrm{ind}_F(\varphi, \Omega, Y_0)$. Replacing first Ω by Ω_0 if necessary, we may thus assume without loss of generality that φ has relatively compact range.

Assume first in addition that Y_0 has finite dimension. Then we may assume that Ω_0 is bounded and as in the proof of Theorem 6, we find a continuous map $G : \overline{\Omega}_0 \to Y_0$ with

$$\mathrm{ind}_F(G \circ F, \Omega_0, Y_0) = \mathrm{ind}_F(\varphi, \Omega, Y_0) \tag{2}$$

whenever ind_F is an index which is homotopy invariant and satisfies the excision property. If ind_F satisfies also the normalization, the left-hand side of (2) is always the (unique) Brouwer degree $\mathrm{Deg}(G, \Omega_0, 0)$, and so $\mathrm{ind}_F(\varphi, \Omega, Y_0)$ is uniquely determined.

Assume now that $Y_0 = Y$. As in the proof of Theorem 6, we find a finite-dimensional $\widehat{Y} \subseteq Y$ and a continuous bounded $\hat{\varphi} : F^{-1}(\overline{\Omega}_{\widehat{Y}}) \to \widehat{Y}$ with $\Omega_{\widehat{Y}} := \Omega_0 \cap \widehat{Y}$ such that

$$\mathrm{ind}_F(\varphi, \Omega, Y) = \mathrm{ind}_F\big(\hat{\varphi}, \Omega_{\widehat{Y}}, \widehat{Y}\big)$$

for each index ind_F with the required properties. Hence, ind_F is unique for $Y_0 = Y$.

Now we consider the general case of an ANR $Y_0 \subseteq Y$. By Lemma 3, we find some neighborhood $U \subseteq X$ of $F^{-1}(\overline{\Omega})$ and a continuous extension $\varphi : U \to Y_0$ of φ with relatively compact range. Excluding the closed set $\{x \in X \setminus F^{-1}(\Omega) \colon F(x) = \varphi(x)\}$ from U if necessary, we may assume that $F(x) \neq \varphi(x)$ for $x \in U \setminus F^{-1}(\overline{\Omega})$. Since F is proper and continuous, F^{-1} is upper semicontinuous. In particular, there is some open $\Omega_1 \subseteq Y$ with $\overline{\Omega} \subseteq \Omega_1$ and $F^{-1}(\Omega_1) \subseteq U$. Since Y is normal, there is some open $\Omega_2 \subseteq Y$ with $\overline{\Omega} \subseteq \Omega_2$ and $\overline{\Omega}_2 \subseteq \Omega_1$. Then $\mathrm{Coin}(F, \varphi, \overline{\Omega}_2) \subseteq \Omega$. Hence, if ind_F satisfies the permanence and excision property, we must have

$$\mathrm{ind}_F(\varphi, \Omega_2, Y) = \mathrm{ind}_F(\varphi, \Omega_2 \cap Y_0, Y_0) = \mathrm{ind}_F(\varphi, \Omega, Y_0).$$

Since we already proved that the left-hand side is uniquely determined by the properties of ind_F, the claim follows. □

3. Fundamentally restrictible maps

In this section we collect some results of independent interest which are needed in the later sections. For a multivalued map $F: X \to 2^Y$, we use the notations $F(A) := \bigcup\{F(x):\ x \in A\}$, and $F^{-1}(B) := \{x \in X:\ F(x) \subseteq B\}$.

Definition 4. Let X be a topological space, Y be a subset of some topological vector space Z, and $F: X \to 2^Y$. All topological notions refer to the vector space Z. Given $O \subseteq Y$, $H: I \times F^{-1}(O) \to 2^Z$ (with some set I), and sets $V \subseteq Y$ and $\emptyset \neq A \subseteq Z$, we call a set $U \subseteq Y$ (A, V)-*fundamental* (for H on O) if the following holds.

(a) U is convex and closed (in Z) and contains V.
(b) $H(I \times F^{-1}(U \cap O)) + A \subseteq U$.
(c) The relation $F(x) \subseteq O \cap \mathrm{conv}((H(t,x) + A) \cup U)$ implies $F(x) \subseteq U$.

We say that H is (A, V)-*fundamentally restrictible* (on O to U), if U is a (A, V)-fundamental set with compact

$$\overline{\mathrm{conv}}\big(H\big(I \times F^{-1}(U \cap O)\big) + A\big).$$

If $A = \{0\}$ and $V = \emptyset$, we call U *fundamental* respectively H *fundamentally restrictible*.

For $F = \mathrm{id}$ and $O = F(X)$, the notion of fundamental sets is a classical tool to define a degree for noncompact maps; it was apparently first introduced in [47] (see also [33]) and later developed by V.V. Obukhovskiĭ and others, even in the context of multivalued maps [9,11]; for acyclic multivalued maps, see also [10,31,37,48].

Proposition 5. *Suppose that*

$$\overline{\mathrm{conv}}\big(\big(H\big(I \times F^{-1}(O)\big) + A\big) \cup V\big) \subseteq Y. \tag{3}$$

Then H is (A, V)-fundamentally restrictible if for any $U \subseteq Y$ the relation

$$\overline{\mathrm{conv}}\big(\big(H\big(I \times F^{-1}(U \cap O)\big) + A\big) \cup V\big) = U \tag{4}$$

implies that U is compact.

Proof. Let U be the intersection of all (A, V)-fundamental sets. Then U is (A, V)-fundamental and satisfies (4). □

In order to apply Proposition 5 it is often convenient to consider only countable sets instead of U which can be done using one of the following two results.

Proposition 6. *Let in the situation of Definition* 4 *the space Z be metrizable, and F be single-valued. Assume that $\overline{\mathrm{conv}}\,A$ and $\overline{\mathrm{conv}}\,V$ are compact and that $H(I \times \{x\})$ is separable for each $x \in F^{-1}(O)$. Let $U \subseteq Y$ satisfy* (4)*. Assume that for any countable $C \subseteq F^{-1}(U \cap O)$ the relation*

$$O \cap F(X) \cap \mathrm{conv}\bigl(\bigl(H(I \times C) + A\bigr) \cup V\bigr) \subseteq \overline{F(C)} \subseteq \overline{O \cap F(X) \cap \overline{\mathrm{conv}}\bigl(\bigl(H(I \times C) + A\bigr) \cup V\bigr)} \tag{5}$$

implies that $\overline{\mathrm{conv}}(H(I \times C))$ is compact. Then U is compact.

If $U \setminus O$ is closed, one may equivalently replace (5) *by*

$$\overline{F(C)} = \overline{O \cap F(X) \cap \mathrm{conv}\bigl(\bigl(H(I \times C) + A\bigr) \cup V\bigr)}. \tag{6}$$

Concerning the following separability assumption, note that $H(I \times \{x\})$ is separable, if I is metrizable and separable and $H(\cdot, x)$ is upper semicontinuous with separable $H(t, x)$ [42].

Proposition 7. *Let in the situation of Definition* 4 *the space Z be metrizable. Let $U \subseteq Y$ satisfy* (4)*, and $G_1, G_2, \ldots \subseteq Z$ be* (*at most*) *countably many sets such that $G_n \cap ((H(I \times F^{-1}(\{u\})) + A) \cup V)$ is separable for each $u \in U$ and each n. Then for any countable $C_0 \subseteq U$ there is some countable $C \subseteq U$ which contains C_0 and satisfies*

$$C \subseteq \overline{\mathrm{conv}}\bigl(\bigl(H\bigl(I \times \bigl(F^{-1}(C \cap O)\bigr)\bigr) + A\bigr) \cup V\bigr), \tag{7}$$

$$G_n \cap \mathrm{conv}\bigl(\bigl(H\bigl(I \times \bigl(F^{-1}(C \cap O)\bigr)\bigr) + A\bigr) \cup V\bigr) \subseteq \overline{G_n \cap C} \quad (n = 1, 2, \ldots). \tag{8}$$

In particular, if for any countable $C \subseteq U$ satisfying (7) *and* (8) *the set $\overline{C}$ is compact, then U is compact.*

Proposition 6 follows from the following result with $D := F^{-1}(U)$ and $W := F^{-1}(O)$.

Proposition 8. *Let X be some set, Z a metric vector space, and $F : X \to Z$. Let $A, V \subseteq Z$, $D, W \subseteq X$, I be some set, and $H : I \times (D \cap W) \to 2^Z$. Assume that A, V and each $H(I \times \{x\})$ are separable, and*

$$F^{-1}\bigl(\overline{\mathrm{conv}}\bigl(\bigl(H\bigl(I \times (D \cap W)\bigr) + A\bigr) \cup V\bigr)\bigr) = D. \tag{9}$$

Then for any countable $C_0 \subseteq D \cap W$ there is some countable $C \subseteq D \cap W$ which contains C_0 and satisfies

$$F(W) \cap \mathrm{conv}\bigl(\bigl(H(I \times C) + A\bigr) \cup V\bigr) \subseteq \overline{F(C)} \subseteq \overline{F(W) \cap \overline{\mathrm{conv}}\bigl(\bigl(H(I \times C) + A\bigr) \cup V\bigr)}. \tag{10}$$

In particular, if $\overline{\mathrm{conv}}\,A$ and $\overline{\mathrm{conv}}\,V$ are compact and any countable $C \subseteq D \cap W$ with (10) *has the property that $\overline{\mathrm{conv}}(H(I \times C))$ is compact, then $\overline{\mathrm{conv}}((H(I \times (D \cap W)) + A) \cup V)$ is compact.*

If $\overline{\mathrm{conv}}((H(I \times (D \cap W)) + A) \cup V) \setminus F(W)$ is closed, one may replace (10) *equivalently by*

$$\overline{F(C)} = \overline{F(W) \cap \mathrm{conv}\bigl(\bigl(H(I \times C) + A\bigr) \cup V\bigr)}. \tag{11}$$

The proofs of Proposition 7 and 8 can be obtained as similar results in [42] and are skipped.

4. Sufficient conditions in the noncompact case

Throughout this section, let X be a metric space, Y be a closed convex subset of a locally convex metric space Z, and $F: X \to Y$ be continuous and proper and such that a coincidence point index satisfying the axioms from [43] (see also [5]) (including the excision property) exists. In view of Theorem 1, this is in particular the case, if F is a Vietoris map. In the following, all topological notions refer to the space Y.

Definition 5. Let $\Omega \subseteq Y$ be open. Then we call a homotopy $H: [0,1] \times F^{-1}(\overline{\Omega}) \to Y$ *weakly admissible* on Ω, if there is some open $\Omega_0 \subseteq Y$ which contains $\operatorname{Coin}(F, H, \overline{\Omega})$ and such that H is fundamentally restrictible on Ω_0. Similarly, we call a continuous map $\varphi: F^{-1}(\overline{\Omega}) \to Y$ *weakly admissible*, if the constant homotopy $H(t,\cdot) := \varphi$ has this property.

The above definition of weakly admissible maps is actually equivalent to the one given in [43]:

Proposition 9. *Let H be weakly admissible on Ω. Then there is some open $\Omega_0 \subseteq Y$ which contains* $\operatorname{Coin}(F, H, \overline{\Omega})$ *and such that H is fundamentally restrictible on $\overline{\Omega}_0$.*

By the results in [43], the given index may thus be extended such that $\operatorname{ind}_F(\varphi, \Omega, Y)$ is defined whenever φ is weakly admissible. This index has analogous properties to those of Theorem 1. In particular, it is invariant under weakly admissible homotopies and has the coincidence point and permanence properties. Moreover, it satisfies the excision property (and is even additive if the given index was additive). For details, we refer to [43] (see also [5]).

It has been observed in [44] that even for $F = \mathrm{id}$ in a Banach space $Y = Z$ it may happen in simple cases that there are weakly admissible maps φ with $\operatorname{ind}_{\mathrm{id}}(\varphi, \Omega, Y) \neq 0$ such that id fails to be φ-coepi on Ω. Therefore, we have to require slightly more.

Definition 6. Let $Y = Z$, and $\Omega \subseteq Y$ open. We call a continuous map $\varphi: F^{-1}(\overline{\Omega}) \to Y$ *coepi-admissible* on Ω, if $\operatorname{Coin}(F, \varphi, \partial\Omega) = \emptyset$ and if φ is $(A, \emptyset)$-fundamentally restrictible on Ω for any convex compact set $A \subseteq Z$.

In particular, any coepi-admissible map is weakly admissible.

Theorem 7 (Sufficient criterion for being coepi). *Let $Y = Z$, and φ be coepi-admissible on Ω with* $\operatorname{ind}_F(\varphi, \Omega, Y) \neq 0$. *Then F is convexly φ-coepi on Ω.*

Proof. Put $\widehat{X} := F^{-1}(\overline{\Omega})$, and let a continuous map $\psi: \widehat{X} \to Y$ be given with compact $A_0 := \overline{\operatorname{conv}}((\psi - \varphi)(F^{-1}(\Omega)))$ such that $\psi(x) = \varphi(x)$ on $B := F^{-1}(\partial\Omega)$. We have to prove that $F(x) = \psi(x)$ has a solution in $F^{-1}(\Omega)$. Since $A := \overline{\operatorname{conv}}(A_0 \cup \{0\})$ is compact,

φ is $(A, \emptyset)$-fundamentally restrictible on Ω to some set U. Then U is fundamental on Ω for the homotopy $H(\lambda, \cdot) := \varphi + \lambda(\psi - \varphi)$ as follows from

$$H\big([0,1] \times F^{-1}(U \cap \Omega)\big) \subseteq \operatorname{conv}\big(\varphi\big(F^{-1}(U \cap \Omega)\big) + A\big) \subseteq \overline{\operatorname{conv}}\, U = U,$$
$$\operatorname{conv}\big(\big\{H(\lambda, x)\big\} \cup U\big) \subseteq \operatorname{conv}\big(\operatorname{conv}\big(\varphi(x) + A\big) \cup U\big) = \operatorname{conv}\big(\big(\varphi(x) + A\big) \cup U\big).$$

Since $\overline{\operatorname{conv}}(H([0,1] \times F^{-1}(U \cap \Omega))) \subseteq \overline{\operatorname{conv}}(\varphi(F^{-1}(U)) + A)$ is compact, H is fundamentally restrictible to U, and thus weakly admissible in view of $\operatorname{Coin}(F, H, \partial\Omega) = \operatorname{Coin}(F, \varphi, \partial\Omega) = \emptyset$. The homotopy invariance of the index thus implies

$$\begin{aligned} \operatorname{ind}_F(\psi, \Omega, Y) &= \operatorname{ind}_F\big(H(1, \cdot), \Omega, Y\big) = \operatorname{ind}_F\big(H(0, \cdot), \Omega, Y\big) \\ &= \operatorname{ind}_F(\varphi, \Omega, Y) \neq 0, \end{aligned}$$

and so $\operatorname{Coin}(F, \psi, \Omega) \neq \emptyset$ by the coincidence point property. □

5. Condensing pairs

The notion of a condensing map is a well-known concept for the degree theory of noncompact maps (see, e.g., [1,14,38]). In this section, we introduce corresponding definitions for pairs. Throughout, let $Y = Z$ be a locally convex metrizable vector space, and X be a metric space.

Definition 7. Let Γ be the class of all functions γ which are defined on a system $\operatorname{dom}\gamma$ of subsets of Z with values in the positive cone of some partially ordered vector space (which may depend on γ) and the following properties.

(a) If $M \in \operatorname{dom}\gamma$, then $\overline{\operatorname{conv}}\, M \in \operatorname{dom}\gamma$ and $\gamma(M) = \gamma(\overline{\operatorname{conv}}\, M)$.
(b) If $M \in \operatorname{dom}\gamma$ and $N \subseteq M$, then $N \in \operatorname{dom}\gamma$ and $\gamma(N) \leqslant \gamma(M)$.
(c) If $M \in \operatorname{dom}\gamma$ and $A \subseteq Z$ is compact and convex, then $M + A \in \operatorname{dom}\gamma$ and $\gamma(M + A) = \gamma(M)$.
(d) If $M \in \operatorname{dom}\gamma$ and $V \subseteq Z$ is compact, then $M \cup V \in \operatorname{dom}\gamma$ and $\gamma(M \cup V) = \gamma(M)$.

Let Γ^* be the class of all $\gamma \in \Gamma$ with the following additional properties.

(e) If $M, N \in \operatorname{dom}\gamma$ and $M + N \subseteq Z$, then $M + N \in \operatorname{dom}\gamma$ and $\gamma(M + N) \leqslant \gamma(M) + \gamma(N)$.
(f) If $M \in \operatorname{dom}\gamma$ and $\lambda M \subseteq Z$ for some $\lambda \in \mathbb{R}$, then $\lambda M \in \operatorname{dom}\gamma$ and $\gamma(\lambda M) = |\lambda|\gamma(M)$.

The above properties are satisfied if γ is a measure of noncompactness in the sense of Sadovskiĭ which is monotone, nonsingular, additive, algebraic additive and homogeneous [38] (see also, e.g., [1]); in this case $\operatorname{dom}\gamma$ is the system of all bounded subsets of Z. The most important examples for such measures of noncompactness are the following ones.

Example 1. Recall that the *Hausdorff measure of noncompactness* $\chi(M)$ of a set $M \subseteq Z$ is the infimum of all $\varepsilon > 0$ such that M has a finite ε-net. Similarly, the *Kuratowski measure of noncompactness* $\alpha(M)$ is the infimum of all $\delta > 0$ such that M can be covered by finitely many sets of diameter less than δ. If Z is normed, then $\chi, \alpha \in \Gamma^*$ when we define the domain of χ and α as the system of all bounded subsets of Z.

Example 2. Let $\|\cdot\|_1, \|\cdot\|_2, \ldots$ be a countable family of seminorms which generates the topology of Z. Let B be the system of all bounded subsets of Z, s be the space of all real sequences equipped with the pointwise order, and $\gamma : B \to s$ be defined by $\gamma(M) := (\gamma_1(M), \gamma_2(M), \ldots)$ where $\gamma_k(M)$ denotes either the Hausdorff or the Kuratowski measure of noncompactness with respect to the seminorm $\|\cdot\|_k$. Then $\gamma \in \Gamma^*$.

Let $O \subseteq X$ and $H : I \times F^{-1}(O) \to 2^Z$.

Definition 8. We call the pair (F, H) *condensing* on O (with respect to Z) if for any $C \subseteq F^{-1}(O)$ for which $\overline{\operatorname{conv}}(H(I \times C))$ is not compact there is some $\gamma \in \Gamma$ with $H(I \times C) \in \operatorname{dom}\gamma$ such that either

(a) $F(C) \notin \operatorname{dom}\gamma$, or
(b) $\gamma(F(C)) \nleq \gamma(H(I \times C))$.

We call (F, H) *k-condensing* on O, if for any $C \subseteq F^{-1}(O)$ for which $\overline{\operatorname{conv}}(H(I \times C))$ is not compact, there is some $\gamma \in \Gamma^*$ with $H(I \times C) \in \operatorname{dom}\gamma$ such that either

(a) $F(C) \notin \operatorname{dom}\gamma$, or
(b) $k\gamma(F(C)) \nleq \gamma(H(I \times C))$.

We call (F, H) *strictly condensing* if it is countably k-condensing for some $k \in [0, 1)$.

If the above assumptions hold at least for countable sets $C \subseteq F^{-1}(O)$, we call (F, H) *countably condensing*, etc. If we can always choose the same function γ (independent of the set C), we call (F, H) *condensing with respect to* γ, etc. We use analogous notions for pairs (F, φ) with $\varphi : F^{-1}(O) \to 2^Y$ and also for single-valued maps.

Clearly, each 1-condensing pair is condensing. The converse need not hold because $\Gamma^* \subseteq \Gamma$.

Example 3. Let Z be a Banach space, $Y \subseteq Z$ be closed and convex, and let α denote the Kuratowski measure of noncompactness. Let $\varphi : F^{-1}(O) \to Y$ and assume that there are constants $\ell > 0$ and $L < \infty$ with

$$\alpha\big(F(C)\big) \geqslant \ell\alpha(C), \tag{12}$$

$$\alpha\big(\varphi(C)\big) \leqslant L\alpha(C) \tag{13}$$

for each countable $C \subseteq F^{-1}(O)$. Condition (12) means that F is “uniformly proper” in some sense, and condition (13) is satisfied, for example, if φ is a compact perturbation of a Lipschitz map with constant L. If O is bounded, then (F, φ) is countably k-condensing with respect to α for each $k > L/\ell$. In particular, if $\ell > L$ and O is bounded, then (F, φ) is strictly countably condensing with respect to α.

Using Proposition 5 and 6, one can prove:

Proposition 10. *If (F, H) is countably condensing on O, then H is (A, V)-fundamentally restrictible on O for any compact convex set $A, V \subseteq Y$.*

If we are only interested in the case $V = \emptyset$, we may of course drop the requirement (d) of Definition 7. In particular, we may drop this requirement for the following two consequences.

Corollary 1. *Let $\Omega \subseteq Y = Z$ be open, and $H : [0, 1] \times F^{-1}(\overline{\Omega}) \to Y$ be continuous. If (F, H) is countably condensing on Ω and* $\mathrm{Coin}(F, H, \partial\Omega) = \emptyset$*, then (F, H) is coepi-admissible.*

Theorem 8 (Sufficient criterion for being coepi). *Let $\Omega \subseteq Y$ be open, and $\varphi : F^{-1}(\overline{\Omega}) \to Y$ be continuous such that (F, φ) is countably condensing on Ω and* $\mathrm{Coin}(F, \varphi, \partial\Omega) = \emptyset$*. Then* $\mathrm{ind}_F(\varphi, \Omega, Y)$ *is defined, and if it is nonzero, then F is convexly φ-coepi on Ω.*

We could have obtained Theorem 8 also in a different way by considering weakly admissible *homotopies* instead of coepi-admissible maps.

6. Strictly coepi maps

Throughout this section, let X be a metric space, Z be a locally convex metrizable space, $Y \subseteq Z$ closed, and $F : X \to Y$. We fix some $\gamma \in \Gamma^*$. The following notion generalizes $(0, k)$-epi maps which have been introduced in [41] (see also [30]).

Definition 9. Let $\Omega \subseteq Y$, and $f : F^{-1}(\overline{\Omega}) \to Y$ be such that F is f-coadmissible on Ω. Given $k \geqslant 0$, we call F (*countably*) *convexly* (f, k, γ)*-coepi* (on Ω with respect to (Y, Z)) if for each continuous $\varphi : F^{-1}(\overline{\Omega}) \to Y$ for which $(F, \varphi - f)$ is (countably) k-condensing with respect to γ on $\overline{\Omega}$ and which satisfies $\varphi(x) = f(x)$ on $F^{-1}(\partial\Omega)$, we have $\mathrm{Coin}(F, \varphi, \Omega) \neq \emptyset$.

We call F *strictly* (*countably*) *convexly* f*-coepi* (on Ω with respect to (Y, Z) and γ) if it is (countably) convexly (f, k, γ)-coepi for some $k > 0$.

Clearly, each convexly (f, k)-coepi map is convexly f-coepi. In particular, each strictly convexly f-coepi map is convexly f-coepi. The following deep result states that the converse holds under mild compactness assumptions. Recall that $Y \subseteq Z$ is called a *cone*, if Y is closed and convex with $0 \in Y + Y \subseteq Y$.

Theorem 9 (Coepi maps F are strictly coepi if (F, f) is condensing). *Let $Y \subseteq Z$ be a cone, $\Omega \subseteq Y$, and $F: X \to Y$ be continuous on $F^{-1}(\partial\Omega)$. Let $\gamma \in \Gamma^*$ assume values in a totally ordered space.*

If F is convexly f-coepi on Ω (with respect to (Y, Z)) and (F, f) is countably k-condensing on Ω with respect to γ for some $k < 1$, then F is countably convexly $(f, 1-k, \gamma)$-coepi on Ω (with respect to (Y, Z)).

In particular, if F is convexly f-coepi and (F, f) is strictly countably condensing with respect to γ, then F is strictly convexly f-coepi with respect to γ.

We postpone the proof to Section 7. We need the following generalizations of Propositions 2–4.

Proposition 11 (Homotopic stability). *Assume that $F: X \to Y$ is continuous and (countably) convexly (f, k, γ)-coepi on $\Omega \subseteq Y$. Let $H: [0,1] \times F^{-1}(\overline{\Omega}) \to Y$ be continuous with $H(0,\cdot) = 0$, and $F(x) - H(t,x) \neq f(x)$ on $[0,1] \times F^{-1}(\partial\Omega)$.*

(a) *If (F, H) is (countably) k-condensing with respect to γ on Ω, then for each continuous $\varphi: F^{-1}(\overline{\Omega}) \to Y$ with $\varphi(x) = f(x)$ on $F^{-1}(\partial\Omega)$ with compact $\overline{\operatorname{conv}}((\varphi - f)(F^{-1}(\Omega)))$, the equation $F(x) - H(1,\cdot) = \varphi(x)$ has a solution in $F^{-1}(\Omega)$.*

(b) *If (F, H) is (countably) k_0-condensing with respect to γ on Ω for some $k_0 \in [0, k]$, then for each continuous $\varphi: F^{-1}(\overline{\Omega}) \to Y$ with $\varphi(x) = f(x)$ on $F^{-1}(\partial\Omega)$ for which $(F, \varphi - f)$ is (countably) $(k - k_0)$-condensing with respect to γ on Ω, the equation $F(x) - H(1,\cdot) = \varphi(x)$ has a solution in $F^{-1}(\Omega)$, provided that γ assumes its values in a totally ordered space.*

Sketch of proof. By Urysohn, there is a continuous function $\lambda: F^{-1}(\overline{\Omega}) \to [0,1]$ which vanishes on $F^{-1}(\partial\Omega)$ and is 1 on the solution set of $F(x) = \Phi(x)$ where $\Phi(x) := \varphi(x) + H(\lambda(x), x)$. Some calculations show that $(F, \Phi - f)$ is (countably) k-condensing with respect to γ on Ω, and so $F(x) = \Phi(x)$ has a solution. $\square$

Proposition 12 (Restriction property). *Assume that F is (countably) convexly (f, k, γ)-coepi on Ω. If $\Omega_0 \subseteq \Omega$ with some open $\Omega_0 \subseteq Y$ such that $\operatorname{Coin}(F, f, \overline{\Omega} \setminus \Omega_0) = \emptyset$, then F is (countably) convexly (f, k, γ)-coepi on Ω_0.*

Proposition 13 (Restriction property for components). *Let $\Omega \subseteq Y$ be open and such that all components of Ω are also open (e.g., Y is locally connected). Assume that F is continuous and (countably) convexly (f, k, γ)-coepi on Ω where f is continuous. If $\operatorname{Coin}(F, f, \Omega)$ is contained in a compact set, then there is some component Ω_0 of Ω such that F is (countably) convexly (f, k, γ)-coepi on Ω_0.*

7. Necessary conditions for the noncompact case

In this section, we use the methods from [20] to generalize Theorem 3 for noncompact maps φ. The main tool used in [20] is the result from [45] which states that 0-epi maps are stable under certain noncompact homotopic perturbations if certain other compactness assumptions are satisfied. We will apply this result in the variant of [20, Theorem 2.2]. Let us first use this result to provide the postponed proof (which could also be obtained from the result in [45]).

Sketch of the proof of Theorem 9. We apply [20, Theorem 2.2] with $D := F^{-1}(\overline{\Omega})$, $B := F^{-1}(\partial\Omega)$, $O := F^{-1}(\Omega)$, $G := F - f$, $K := Y$, and $Y := Z$. Since F is convexly f-coepi and $Y + Y \subseteq Y$, it can be verified that the map G has the coincidence point property required for that result.

Now let some continuous function $g : D \to Y$ be given with $g(x) = f(x)$ on B such that $(F, g - f)$ is $(1 - k)$-condensing with respect to γ. We have to prove that $F(x) = g(x)$ has some solution in O. To this end, it suffices to prove that the assumptions of [20, Theorem 2.2] are satisfied with $H(\lambda, \cdot) := \lambda(g - f)$, because this result then implies that there is some $x \in O$ with $G(x) + H(1, x) = 0$ which means $F(x) = g(x)$, as required. These assumptions are already satisfied for the trivial partition $t_0 = 0$ and $t_1 = 1$ with $U := D$ and $V := \{0\}$. This can be verified by some calculations based on the estimate $\gamma(F(C)) \leqslant \gamma(G(C)) + \gamma(f(C))$. □

For the rest of this section, let X be a metric space, $Y = Z$ be a metrizable locally convex space, $F : X \to Y$ be a Vietoris* map, $\Omega \subseteq Y$ be open, $D := F^{-1}(\overline{\Omega})$, and $\varphi : F^{-1}(\overline{\Omega}) \to Y$ be continuous with $\operatorname{Coin}(F, \varphi, \partial\Omega) = \emptyset$.

Lemma 4. *Let φ be fundamentally restrictible on Ω. Then there is some open set $\Omega_0 \subseteq \Omega$ which contains* $\operatorname{Coin}(F, \varphi, \overline{\Omega})$ *and some continuous function $\varphi_0 : D \to Y$ with compact* $\overline{\operatorname{conv}}(\varphi_0(D))$ *such that the convex homotopy $h(\lambda, x) := (1 - \lambda)\varphi(x) + \lambda\varphi_0(x)$ satisfies* $\operatorname{Coin}(F, h, \partial\Omega_0) = \emptyset$. *If Ω is connected, it may also be arranged that Ω_0 is connected.*

Sketch of proof. Observe that $C := \operatorname{Coin}(F, \varphi, \overline{\Omega})$ is closed and contained in Ω. Let U be a compact fundamental set for φ on Ω. Then $C \subseteq U$, and so C is actually compact. Since Y is normal, we find some open $\Omega_0 \supseteq C$ with $\overline{\Omega}_0 \subseteq \Omega$. In view of Lemma 2, we may assume that Ω_0 is connected if Ω is connected. Since U is fundamental on Ω, we have in particular:

(a) $\varphi(F^{-1}(U \cap \overline{\Omega}_0)) \subseteq U$.
(b) $F(x) \in \overline{\Omega}_0 \cap \operatorname{conv}(\{\varphi(x)\} \cup U)$ implies $F(x) \in U$.

Moreover, U is convex and compact, and so there is some (continuous) retraction $r : Y \to U$, i.e., $r|_U = \mathrm{id}$. This can be proved even without the (uncountable) axiom of choice (see, e.g., [42, Lemma 1.3]). We claim that $\varphi_0 := r \circ \varphi$ has the required properties.

Indeed, assume contrary that there is some $x \in F^{-1}(\partial\Omega_0)$ and some $\lambda \in [0, 1]$ with $F(x) = (1 - \lambda)\varphi(x) + \lambda\varphi_0(x)$. Then $F(x) \in \overline{\Omega}_0 \cap \operatorname{conv}(\{\varphi(x)\} \cup U)$, and so $F(x) \in U$, i.e., $x \in F^{-1}(U \cap \overline{\Omega}_0)$ which in turn implies $\varphi(x) \in U$, and so $\varphi_0(x) = \varphi(x)$ and thus $F(x) = (1 - \lambda)\varphi(x) + \lambda\varphi(x) = \varphi(x)$ which contradicts $\operatorname{Coin}(F, \varphi, \partial\Omega_0) = \emptyset$. □

If (F,φ) is countably condensing on Ω, then φ is fundamentally restrictible on Ω (Proposition 10). In particular, φ is weakly admissible, and the index $\operatorname{ind}_F(\varphi,\Omega,Y)$ is defined as described in Section 4. Then the essential result reads as follows.

Theorem 10. *Let (F,φ) be strictly countably condensing on Ω. If $\operatorname{ind}_F(\varphi,\Omega,Y)=0$ and Ω is connected, then F is not convexly φ-coepi on Ω.*

Sketch of proof. Let Ω_0, φ_0, and h be as in Lemma 4. The excision property of the index implies $\operatorname{ind}_F(\varphi,\Omega_0,Y)=0$. Note that we have for any $M\subseteq D:=F^{-1}(\Omega_0)$ that $\overline{\operatorname{conv}}(h([0,1]\times M))=\overline{\operatorname{conv}}(\varphi(M)\cup\varphi_0(M))$. Then (F,h) is countably condensing on Ω, and by homotopy invariance $\operatorname{ind}_F(\varphi_0,\Omega_0,Y)=\operatorname{ind}_F(\varphi,\Omega_0,Y)=0$. By Theorem 3 (or Theorem 6), we find in particular some continuous function $f_0:D\to Y$ with compact $\overline{\operatorname{conv}}(f_0(D))$ such that for $B:=F^{-1}(\partial\Omega_0)$ we have $f_0|_B=\varphi_0|_B$ and such that $\operatorname{Coin}(F,f_0,\Omega_0)=\emptyset$. In particular, for the function $f=f_0-\varphi_0$ we have compact $\overline{\operatorname{conv}}(f(D))$ and $f|_B=0$, but $F(x)-\varphi_0(x)=f(x)$ has no solution in $O:=F^{-1}(\Omega_0)$. This means that the conclusion of [20, Theorem 2.2] fails for the function $G:=F-\varphi$ and the homotopy $H(\lambda,x):=\lambda(\varphi_0(x)-\varphi(x))$.

On the other hand, one can verify that the assumptions of [20, Theorem 2.2] hold if F is convexly φ-coepi on Ω which then proves the claim. Indeed, Proposition 3 and the choice of Ω_0 imply that F is φ-coepi on Ω_0. From this, one can conclude that G has indeed the coincidence point property required for [20, Theorem 2.2]. Also the compactness assumption of [20, Theorem 2.2] can be verified if the partition $0=t_0\leqslant t_1\leqslant\cdots\leqslant t_n=1$ is chosen such that $t_i-t_{i-1}\leqslant 1/k-1$ where $k\in(0,1)$ is such that (F,φ) is countably k-condensing on Ω_0. □

In the case $k=1$, we obtain only a slightly weaker conclusion.

Theorem 11. *Let (F,φ) be* (*countably*) *1-condensing on Ω with respect to some $\gamma\in\Gamma^*$. Assume that $\operatorname{ind}_F(\varphi,\Omega,Y)=0$ and Ω is connected. Then F is not strictly* (*countably*) *convexly φ-coepi on Ω with respect to γ.*

Proof. Assume contrary that F is (countably) convexly (φ,k,γ)-coepi on Ω for some $\gamma\in\Gamma^*$ and some $k\in(0,1)$. We use the notation of the proof of Theorem 10 with the difference that we put $G:=F-\varphi-k(\varphi_0-\varphi)$ and $H(\lambda,\cdot):=\lambda(1-k)(\varphi_0-\varphi)$. Since $G-H(1,\cdot)=F-\varphi_0$ is the same function as in the proof of Theorem 10, the same arguments show that the conclusion of [20, Theorem 2.2] fails. It remains to show that the assumptions of that theorem are satisfied.

By Proposition 12, F is (countably) convexly (φ,k,γ)-coepi on Ω_0. To see the coincidence point property required for G in [20, Theorem 2.2], let some continuous $f:D\to Y$ with $f|_B=0$ and compact $\overline{\operatorname{conv}}(f(D))$ be given. For the homotopy $h_0(\lambda,\cdot):=\lambda k(\varphi-\varphi_0)$, one can show that (F,h_0) is (countably) k-condensing with respect to γ, and so Proposition 11 implies that $G(x)=F(x)-\varphi(x)-h_0(1,x)=f(x)$ has a solution x in $O=F^{-1}(\Omega_0)$, as required. Also the compactness requirement of [20, Theorem 2.2] is satisfied if the partition $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = 1$ is chosen such that $c_i := t_i - t_{i-1} \leqslant 1/(1-k) - 1$. This can be verified using the fact that the function

$$G_i = F - \bigl(1 - k - t_{i-1}(1-k)\bigr)\varphi - \bigl(k + t_{i-1}(1-k)\bigr)\varphi_0$$

satisfies

$$\begin{aligned}\gamma\bigl(F(C)\bigr) &\leqslant \gamma\bigl(G_i(C)\bigr) + \bigl(1 - k - t_{i-1}(1-k)\bigr)\gamma\bigl(\varphi(C)\bigr)\\ &\leqslant (1 + c_i - t_{i-1})(1-k)\gamma\bigl(\varphi(C)\bigr). \qquad \square\end{aligned}$$

Note that Theorem 10 is "almost" a special case of Theorem 11 in view of Theorem 9. The main result of this paper can now be summarized as follows.

Theorem 12 (Hopf extension theorem for the noncompact case). *Let Y be a locally convex metrizable space, X some metric space, and $F : X \to Y$ be a Vietoris* map. Let $\Omega \subseteq Y$ be open, and $\varphi : F^{-1}(\overline{\Omega}) \to Y$ be continuous with* $\operatorname{Coin}(F, \varphi, \partial\Omega) = \emptyset$ *and such that (F, φ) is countably condensing. Then* $\operatorname{ind}_F(\varphi, \Omega_0, Y)$ *is defined for each component Ω_0 of Ω, and the following holds.*

(a) *If (F, φ) is strictly countably condensing, then F is convexly φ-coepi on Ω if and only if there is some component Ω_0 of Ω with* $\operatorname{ind}_F(\varphi, \Omega_0, Y) \neq 0$.
(b) *If (F, φ) is* (*countably*) 1-*condensing with respect to some $\gamma \in \Gamma^*$ and F is strictly* (*countably*) *convexly φ-coepi with respect to γ, then* $\operatorname{ind}_F(\varphi, \Omega_0, Y) \neq 0$ *for some component Ω_0 of Ω.*
(c) *Conversely, if* $\operatorname{ind}_F(\varphi, \Omega_0, Y) \neq 0$ *for some component Ω_0 of Ω, then F is convexly φ-coepi on Ω.*

Note that $F = \mathrm{id}$ in a Banach space is convexly φ-coepi on Ω if and only if $\mathrm{id} - \varphi$ is 0-epi on Ω. Hence, Theorem 12 contains the main theorems from [20] as special cases. Also the following important consequence of Theorem 11 and the excision property of the index generalizes the corresponding corollary of [20] and thus gives a positive partial answer to the question raised there.

Corollary 2 (Excision for coepi maps). *Let X and Y be as in Theorem* 12, *and $F : X \to Y$ be a Vietoris* map. Assume that (F, φ) is strictly countably condensing on Ω. Let $\Omega_0 \subseteq \Omega$ be open and such that each component of Ω contains at most one component of Ω_0. If* $\operatorname{Coin}(F, \varphi, \overline{\Omega} \setminus \Omega_0) = \emptyset$, *then F is convexly φ-coepi on Ω if and only if it is convexly φ-coepi on Ω_0.*

The example given in [20] shows that Corollary 2 fails in any infinite-dimensional Banach space even for $F = \mathrm{id}$, if we drop the assumption that (id, φ) be strictly countably condensing. The classical example $F = \mathrm{id}$ in $X = \mathbb{R}$ and $\varphi(x) = x + (x^2 - 2)$ for $\Omega = (-2, 2)$ and $\Omega_0 = (-2, -1) \cup (1, 2)$ shows that the assumption on the components cannot be dropped.

It remains an open problem to which extent the assumption that F be a Vietoris* map can be weakened: We do not even know whether Corollary 2 holds for Vietoris maps F.

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